SCIENCE CHINA
Technological Sciences

•Article•

https://doi.org/10.1007/s11431-020-1636-7

# Rapid identification of switched systems: A data-driven method in variational framework

LI ChunJiang[1], HUANG ZhiLong[1], WANG Yong[1*] & JIANG HanQing[2]

[1] *Department of Engineering Mechanics, Zhejiang University, Hangzhou 310027, China;*
[2] *School for Engineering of Matter, Transport and Energy, Arizona State University, Tempe AZ 85287, USA*

Received April 10, 2020; accepted May 12, 2020; published online July 23, 2020

Switched systems, i.e., systems changing the parameter values (even structural forms) abruptly and randomly at arbitrary instants, have been extensively utilized in many fields of modern industries. Rapid identification of switched systems, i.e., capturing all the changing instants and reconstructing the mathematical models rapidly, is of great significance for behavior prediction, performance evaluation and possible control, but is restricted by small data amount available. Here, the rapid identification problem is successfully solved by a data-driven method in variational framework. The data-driven method only requires a small amount of data due to the compact form of the variational description, and is robust to data noise due to the holistic viewpoint. Two numerical examples, i.e., Duffing oscillator and van der Pol system (as two representative systems in nonlinear dynamics), are adopted to illustrate its application, efficiency and robustness to noise.

**switched system, rapid identification, data-driven method, small data amount, robustness**

**Citation:** Li C J, Huang Z L, Wang Y, et al. Rapid identification of switched systems: A data-driven method in variational framework. Sci China Tech Sci, 2020, 63, https://doi.org/10.1007/s11431-020-1636-7

## 1 Introduction

In modern industries, a lot of systems change their parameter values and structural forms actively or passively. For instance, a variable-wing aircraft switches the configuration of wings actively to execute various missions [1,2]. A machinery operating in extreme circumstances (e.g., an aircraft exposed to high temperature and strong jet noise) changes its parameter values (e.g., damping and stiffness) passively. The common characteristic of the above systems is that parameter values and structural forms change ceaselessly, rapidly or slowly, continuously or abruptly. Here, we define switched systems explicitly as below: systems with parameter values changing abruptly in time and randomly in amplitude at arbitrary instants, and systems with structural forms changing (e.g., add or delete terms, or response characteristics changes) at arbitrary instants. In accordance with this definition, the time-variable systems [3,4] do not belong to the switched systems. The analytical procedures for the switched systems, however, can be readily generalized to time-variable systems.

The dynamic properties and operating performances of switched systems dramatically vary with time. Thus, it is quite important to grasp the mathematical description (i.e., the governing equations of motion) in real time, which helps us predict behaviors and evaluate performances [5]. Due to the rapidness, randomness and diversity of switches, it is impossible to identify the mathematical description preliminarily in the process of design. There is only one way left, i.e., rapidly identify and reidentify the mathematical description from the captured noisy data.

Due to the abrupt switches of switched systems, there is only small data amount available and only extremely short time to execute the identifying and reidentifying processes.

*Corresponding author (email: ypwang@zju.edu.cn)

 

The rapid development of machine learning and computer technology has given birth to many new algorithms for system identification [6,7]. Most of them, however, tend to be not appropriate to identify switched systems. For instance, neural networks [8–10] have been applied to system identifications for decades, but massive training data and computing time are required in order to gain viable models. Other classical methods (such as Kalman filters [11], eigensystem realization algorithm [12], autoregressive models [13], to name only a few) also provide a prediction of subsequent values of some specific quantities, and cannot derive the analytical expression describing system behaviors. System models identified by all of these methods lack simplicity and interpretability.

Symbolic regression method [14–16] and sparse regression method [17–19] are two recent advances in the field of system identification. These two methods are in the differential framework, i.e., identifying the differential equations of motion by discrete data captured from simulations or experiments. The identified differential equations are simplified by adopting the Pareto criteria [15] and parsimonious rule [15,20] which induce the eminent interpretability. Technically, the symbolic regression method can identify arbitrarily complex nonlinear systems, but, in fact, it is rather prone to overfitting if the parsimony and accuracy are not well balanced, and specially, is extremely time-consuming [14,15]. Thus, it is not reasonable to identify and reidentify switched systems by the symbolic regression method. The sparse regression method has successfully identified the differential equations of state variables of various nonlinear dynamical systems [17], and has been generalized to the rapid model recovery of nonlinear dynamical systems with switched parameters and structures [21]. By combining with the model predictive control strategy, it has been adopted to control nonlinear dynamical systems in the low-data limit [22].

Recently, a data-driven method in variational framework has been established and successfully applied to the variational law identification of physical systems [23]. This data-driven method (i.e., identify variational equations) can be regarded as a counterpart of symbolic regression and sparse regression methods (i.e., identify differential equations). It inherits the advantages of sparse regression method, at the same time has its own advantages. To be specific, the data-driven method only requires a small amount of data due to the compact form of variational description, and is robust to data noise due to the holistic viewpoint (that is to say, the characteristics of integration). Thus, it is appropriate to generalize this data-driven method to rapidly identify and reidentify the variational equations of switched systems. This paper devotes to this subject.

The main structure of this paper is as follow. First, the data-driven methodology of rapidly identifying switched systems is presented in detail. Second, two representative switched systems, i.e., a switched Duffing oscillator with parameter variations and a switched van der Pol system with structure changing, are investigated to demonstrate the application, efficiency and robustness of this data-driven method. Finally, a brief conclusion is given to illustrate the potential applications of the rapid identification method to the real-time control of switched systems.

## 2 Data-driven method to rapidly identify switched systems

For a given switched system, we real-timely capture the discrete data of state variables (and inputs if exist) at discrete instants by various sensors. The time delay induced by sensors can be reduced dramatically by the advances of sensing technologies [24], and so can be omitted. The discrete data captured from experiments are unavoidably contaminated by noises, and so must be discussed in detail. Our objectiveness is to establish a data-driven method, which can identify the variational equation governing the dynamic behaviors only through a small amount of discrete data, and can track the switches of parameter values and structural forms, with fast analysis speed and high robustness to measurement noise.

Here, the data-driven method is established aimed to switched systems with finite degree-of-freedoms. That is to say, these systems are described by ordinarily differential equations or integral-variational equations in temporal domain. The flow chart of the whole process of this method is shown in Figure 1. This whole process consists of the following three successive steps: step 1, capture the discrete data in real time; step 2, identify the variational equation governing the dynamic behaviors of switched system; step 3, detect the occurrence of switches, and reidentify the renewed variational equation once a switch occurs.

### 2.1 Step 1: Capture discrete data in real time

First, we determine the state variables $\mathbf{q}$, $\dot{\mathbf{q}}$ (such as displacements and velocities for mechanical systems, and charges and currents for electric systems), and the excitations $\mathbf{F}$ which may influence the dynamic behaviors (such as external forces for mechanical systems, and voltages for electric systems). Then, we configure the appropriate sensors to detect and record the discrete data of state variables and excitations at discrete instants, i.e., $\{\mathbf{q}(t_i), \dot{\mathbf{q}}(t_i), \mathbf{F}(t_i)\}$. If only $\mathbf{q}(t_i)$ (or $\dot{\mathbf{q}}(t_i)$) are detected by sensors, another quantity $\dot{\mathbf{q}}(t_i)$ (or $\mathbf{q}(t_i)$) must be calculated by various efficient algorithms of numerical differentiation or numerical integration. This data-driven method is based on and only based on the dis-

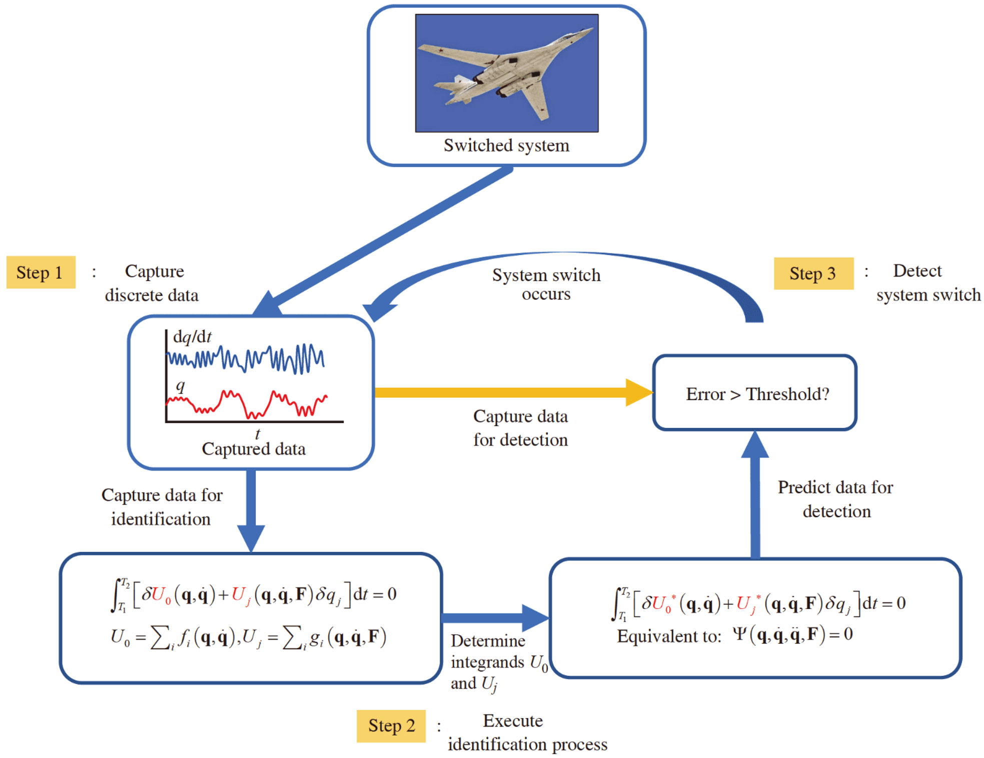


**Figure 1** (Color online) Flow chart of the proposed methodology. The whole process consists of three successive steps: step 1, capture the discrete data in real time; step 2, identify the variational equation governing the dynamic behaviors of the switched system; step 3, detect the occurrence of switches ceaselessly, and reidentify the renewed variational equation once a switch occurs.

crete dataset, not only for identifying and reidentifying the variational equations, but also for detecting the occurrence of switches.

### 2.2 Step 2: Identify the integral-variational equation

For a general physical system, the state variables satisfy the following integral-variational equation [23,25,26]:

$$\int_{T_1}^{T_2}\left[\delta U_0(\mathbf{q},\dot{\mathbf{q}})+\sum_j U_j(\mathbf{q},\dot{\mathbf{q}},\mathbf{F})\delta q_j\right]\mathrm{d}t=0, \tag{1}$$

where $U_0$ and $U_j$ are functions of the state variables (i.e., the motion $\mathbf{q}(t)$ and its time derivative $\dot{\mathbf{q}}(t)$) and the excitations $\mathbf{F}(t)$. $\delta q_j(t)$ are arbitrary variations of the actual motion $\mathbf{q}(t)$ with the constraints at starting and ending instants, that is $\delta q_j(T_1)$=0 and $\delta q_j(T_2)$=0. The integral-variational equation keeps right for arbitrary integral interval $[T_1, T_2]$ and for arbitrary variations $\delta q_j(t)$. The identification of the integral-variational equation is then converted to the identification of the undetermined integrands $U_0$ and $U_j$.

The undetermined integrands $U_0(\mathbf{q},\dot{\mathbf{q}})$ and $U_j(\mathbf{q},\dot{\mathbf{q}},\mathbf{F})$ can be generally expanded on the associated complete bases, respectively, i.e.,

$$U_0(\mathbf{q},\dot{\mathbf{q}})=\sum_{i=1}^{\infty}C_i\cdot f_i(\mathbf{q},\dot{\mathbf{q}}), \tag{2a}$$

$$U_j(\mathbf{q},\dot{\mathbf{q}},\mathbf{F})=\sum_{i=1}^{\infty}D_{j_i}\cdot g_{j_i}(\mathbf{q},\dot{\mathbf{q}},\mathbf{F}). \tag{2b}$$

Here, the functions $f_i$ and $g_{j_i}$ constitute the associated complete bases. For simplicity, they can be directly selected as the power functions of $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\mathbf{F}$, i.e.,

$$f_i(\mathbf{q},\dot{\mathbf{q}})=q_1^{i_1}q_2^{i_2}\cdots q_n^{i_n}\dot{q}_1^{i_{n+1}}\dot{q}_2^{i_{n+2}}\cdots\dot{q}_n^{i_{2n}}, \tag{3a}$$

$$g_{j_i}(\mathbf{q},\dot{\mathbf{q}})=q_1^{j_{i1}}q_2^{j_{i2}}\cdots q_n^{j_{in}}\dot{q}_1^{j_{i(n+1)}}\dot{q}_2^{j_{i(n+2)}}\cdots\dot{q}_n^{j_{i(2n)}}F_1^{j_{i(2n+1)}}F_2^{j_{i(2n+2)}}\cdots F_n^{j_{i(3n)}}, \tag{3b}$$

where $n$ is the degree of freedoms, and $i_s$, $j_{il}$ are non-negative integers. $U_0$ and $U_j$ can be regarded as the linear functions with undetermined coefficients $C_i$ and $D_i$, respectively. Theoretically, the expansions of $U_0$ and $U_j$ should include infinite terms. In practice, however, only the terms with lower-order power are of great significance according to the parsimonious rule [17]. Then, the system identification comes down to the determination of the undetermined coefficients $C_i$ and $D_i$ with finite numbers.

For a given integral interval $[T_1, T_2]$ and a selected variation of motion $\delta\mathbf{q}(t)$, we derive a linear algebraic equation with respect to the undetermined coefficients $C_i$ and $D_i$ by substituting the expansions in eqs. (2a) and (2b) into the integral-variational eq. (1) and then executing the numerical integration by utilizing the captured discrete data. Due to the arbitrary of the integral interval and the variation of motion, we can establish a set of linear algebraic equations with enough numbers to solve the undetermined coefficients $C_i$ and $D_i$. Solving the overdetermined linear algebraic equations by the pseudo-inverse algorithm [27] yields the optimal values of the undetermined coefficients, denoted by $C_i^*$ and $D_i^*$. So far, the integral-variational equation is identified completely, from which we can predict the dynamic behaviors by directly solving the integral-variational equation or solving the associated differential equations.

### 2.3 Step 3: Detect the occurrence of switches and reidentify the renewed variational equation

Once the initial mathematical description of the switched system is obtained from the discrete data, the dynamic behaviors can be easily predicted by numerically solving the integral-variational equation or the associated differential equations with the initial conditions defined by discrete data. The predicted responses are then compared with the actual responses observed by sensors, and the comparison of the predicted and observed responses is executed ceaselessly. Once the deviation between them (according to some definition) exceeds a prescribed threshold, the system is considered to be switched.

There exist many different definitions of deviations, as shown in refs. [28,29]. Here, we adopt an intuitive definition of deviation, i.e., the relative deviation between the predicted trajectory and tested trajectory in a preselected time interval $T$. Sum up the absolute value of the difference between the predicted data $\mathbf{q}_{\text{predicted}}$, $\dot{\mathbf{q}}_{\text{predicted}}$ and observed data $\mathbf{q}_{\text{observed}}$, $\dot{\mathbf{q}}_{\text{observed}}$ at each discrete instant $t_i$, and then divide it by the point numbers in time interval $T$. That is,

$$\text{Error}=\frac{\sum_{i(t_i\in T)}\left[\left|\mathbf{q}_{\text{predicted}}(t_i)-\mathbf{q}_{\text{observed}}(t_i)\right|+\left|\dot{\mathbf{q}}_{\text{predicted}}(t_i)-\dot{\mathbf{q}}_{\text{observed}}(t_i)\right|\right]}{\sum_{i(t_i\in T)}1}. \tag{4}$$

The time interval $T$ to calculate the difference and the threshold to judge the occurrence of switches are initialized at the beginning, and adjusted gradually with the further comprehension on the essential properties of the switched system. At a first glance, if a short time interval and a small threshold are chosen, the procedure will react quickly to the switches. The smaller the time interval and the threshold, the quicker the reaction of the procedure. In fact, for a short time interval the data noise will dramatically influence the value of error, and for a small threshold, the procedure may falsely judge the occurrence of switches. On the contrary, a long-time interval and a large threshold will induce the procedure reacting slowly to the switches. To summarize, adjusting the time interval and the threshold automatically is quite important for the data-driven method to track the switches accurately and timely. Furthermore, different weights can be allocated to the absolute differences of the motion and its time derivative from practical considerations.

Once a switch occurs, we will capture the new discrete data, execute the data-driven procedure again and reidentify the renewed integral-variational equation in eq. (1). That is, we update the undetermined coefficients $C_i$ and $D_i$, the integrands $U_0$ and $U_j$, and then the integral-variational equation governing the present behaviors of the switched system. The close-loop procedure, i.e., the initial identification, detection of deviation, judgement of switches and reidentification, lets us tracking the switched systems in real time.

Here, we provide some comments on the executing time of each step. In the initially-identifying and reidentifying procedures, the integral interval in the integral-variational equation can be arbitrary specified. Thus, we do not need a long-time interval to obtain the linear algebraic equations with enough numbers due to the arbitrary of variations. The time to solve the overdetermined equations can be ignored. The time to evaluate the relative deviation is determined by the length of the preselected time interval, while the time to compare the relative deviation with the threshold can also be ignored. Apparently, the noise in the discrete data influences the effectiveness and efficiency of the data-driven procedure. With the increase of noise intensity, the effective information hidden in data decreases, and the integral interval in the initially-identifying and reidentifying procedures and the preselected time integral in the deviation evaluation should be extended. It is necessary that the efficacy of the data-driven method deteriorates with the increase of noise intensity, and an important work that should be done is to evaluate the robustness of this method to data noise.

## 3 Numerical example 1: Switched Duffing oscillator

From now on, two numerical examples, i.e., Duffing oscillator with parameter switching and van der Pol system with structure switching, are adopted to illustrate the application and efficacy of this data-driven method, and to discuss its robustness to data noise. Two typical excitations, i.e., the harmonic excitation with amplitude and frequency switching abruptly and randomly, and the random excitation described by Gaussian white noise are considered successively. In these numerical examples, the discrete data are generated by numerically simulating the equations of motion with given time step, and some noise described by Gaussian white noise are deliberately added into the exact simulated data to reflect the unavoidable noise induced by sensing and collecting in experiments.

We first identify and reidentify the integral-variational equations of switched Duffing oscillator from the exact discrete data, and then, discuss the influence of data noise on the identifying accuracy and efficiency.

### 3.1 Case I. Duffing oscillator with sinusoidal excitation

A Duffing oscillator with external excitation and system parameter changing satisfies the following differential equation of motion [30]:

$$m\ddot{q} + c\dot{q} + k_1 q + k_3 q^3 = F(t), \tag{5}$$

where $m$ is mass, $c$ linear damping coefficient, $k_1$ and $k_3$ linear and cubic nonlinear stiffness coefficients, respectively, $F$ external excitation. Here, we consider the case with sinusoidal excitation $F(t)=P\sin\omega t$. The damping coefficient $c$, the linear and nonlinear stiffness coefficients $k_1$ and $k_3$, and the amplitude $P$ and circular frequency $\omega$, change abruptly and randomly. The switching values and switching sequences of system parameters are shown in the first row of Figure 2.

Step 1: by numerically simulating eq. (5), we collect the discrete data of excitation and state variables with prescribed sampling frequency (such as 0.001 s here), as shown in the second row of Figure 2. Step 2: identify the integral-variational equation initially from the captured discrete data. Guided by the rule of dimensional consistency [31], we select appropriate base functions of undetermined integrands $U_0$ and $U_1$ (i.e., $q, q^2, \dot{q}^2$ for $U_0$, and $1, F, q, \dot{q}, q^3$ for $U_1$). That is to say, some prior knowledges are included in the selection of base functions. Substituting the linear expansions of $U_0$ and $U_1$ on the associated bases (with coefficients $C_i$ and $D_i$) into eq. (1) and executing the variational calculus yield

$$\int_{T_1}^{T_2} A_1\dot{q}\delta\dot{q} + \left(B_1 + B_2F + B_3q + B_4\dot{q} + B_5q^3\right)\delta q\,\mathrm{d}t = 0, \tag{6}$$

where the undetermined coefficients $A_i$ and $B_i$ are the linear combinations of coefficients $C_i$ and $D_i$. The integral-variational equation of the first stage is identified by using the discrete data in the initial 1 s, as shown in the third row and first column in Figure 2. Step 3: by setting the preselected time interval $T$ (0.5 s here) and the threshold of relative deviation (0.05 here), the identified integral-variational equations of successive stages are shown in the third row in Figure 2.

The exact integral-variational equation of motion of Duffing oscillator is

$$\int_{t_1}^{t_2} m\dot{q}\delta\dot{q} + \left(F + k_1q + c\dot{q} + k_3q^3\right)\delta q\,\mathrm{d}t = 0. \tag{7}$$

It is obvious that the identified integral-variational equation for each stage coincides very well with the exact equation, and the predicted trajectory tracks the observed trajectory accurately and rapidly (with only 1.5 s delay), as shown in the fourth row in Figure 2.

### 3.2 Case II. Duffing oscillator with random excitation

In this data-driven method, there is not any limitation on the

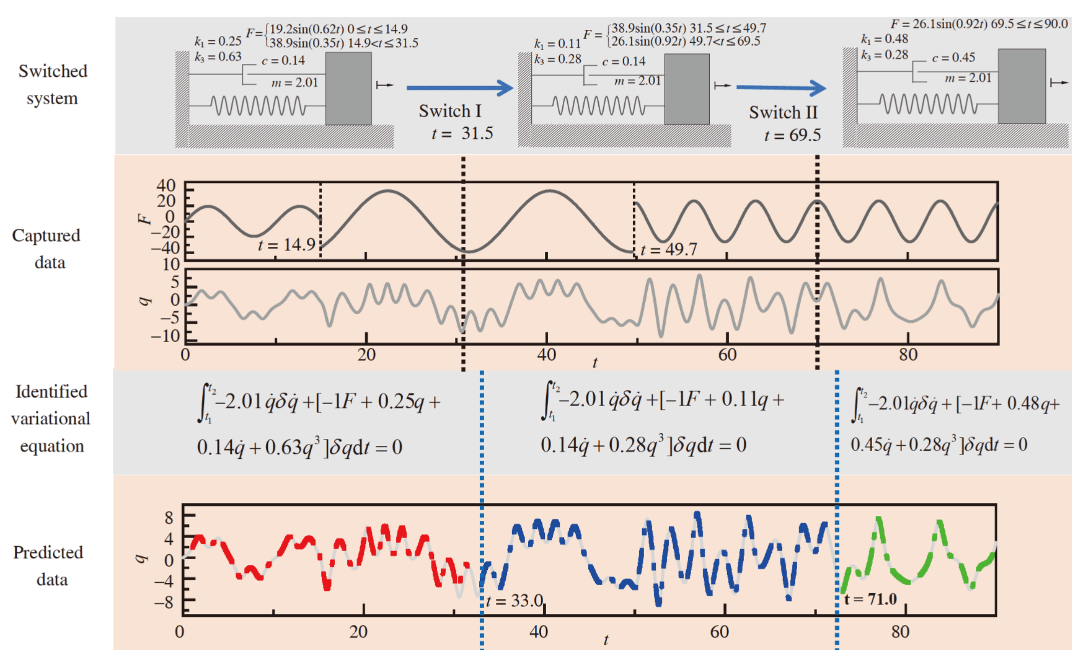


**Figure 2** (Color online) Demonstration of the switched Duffing oscillator with sinusoidal excitation. First row: physical model of Duffing oscillator with preset values of system parameters and moments of switches; second row: captured data of sinusoidal excitation and state variables with sampling frequency 0.001 s; third row: identified variational equation for each stage; fourth row: comparison of predicted trajectory and captured data.

style of excitations. Here, we demonstrate its application and efficacy to rapidly identify the switched Duffing oscillator with random excitation. The random excitation is described by Gaussian white noise $\xi(t)$ [32] with zero mean and excitation intensity $2D$ ($2D$=3.0 here). The switching values and switching sequences of system parameters are the same as above. The discrete data captured from a sample of random excitation, the discrete data of the associated random response and the predicted trajectory from the identified integral-variational equations are depicted in Figure 3. The predicted trajectory tracks the captured data with high precision and a short time delay (4 s here).

### 3.3 Discussions on the influence of data noise: the robustness to data noise

The noise in the captured data will dramatically influence the identifying accuracy and efficiency. The relative deviation between the captured data and the predicted data increase with the intensity of data noise. We should set larger threshold for the captured data with stronger noise, if not, this procedure will get stuck in an endless loop between the judgement of switches and reidentification of integral-variational equation.

Here, the error between the identified/reidentified equations and the exact equations is evaluated by the relative error between the coefficients of integrand terms, that is,

$$e = \frac{\sum_i (A_i - A_i^{\text{exact}})^2 + \sum_j (B_j - B_j^{\text{exact}})^2}{\sum_i (A_i^{\text{exact}})^2 + \sum_j (B_j^{\text{exact}})^2}. \quad (8)$$

The switched Duffing oscillator with sinusoidal excitation is adopted to demonstrate the robustness of this data-driven method to data noise. The relation of the relative error $e$ to the noise intensity of the Gaussian white noise deliberately adding into the exact simulated data are shown in Figure 4, with different integral intervals (5, 6, 8, 12, 30 s) for system identification. The relative error increases monotonically with the noise intensity and decreases monotonically with the prolongation of the integral interval. As the integral interval is over twice of the period of the associated linear system, the improvement of the identifying accuracy becomes not obvious by further prolonging the integral interval. If the integral interval is less than half of this period, it is unlikely to identify the results with acceptable accuracy. Figure 5 depicts the captured noisy data with noise intensity 0.1 and the predicted trajectory identified from the integral interval in 12 s. In this case, the relative error $e$ equals $2.58\times10^{-4}$, and the identifying result coincides very well with the exact result.

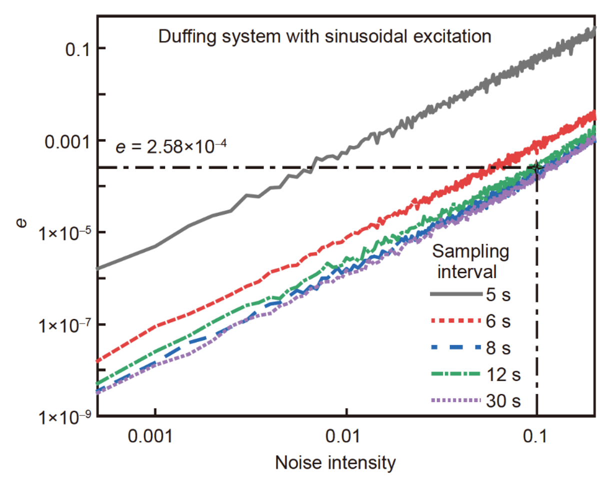


**Figure 4** (Color online) The relation of the relative error of Duffing oscillator with sinusoidal excitation to the noise intensity for five different integral intervals.

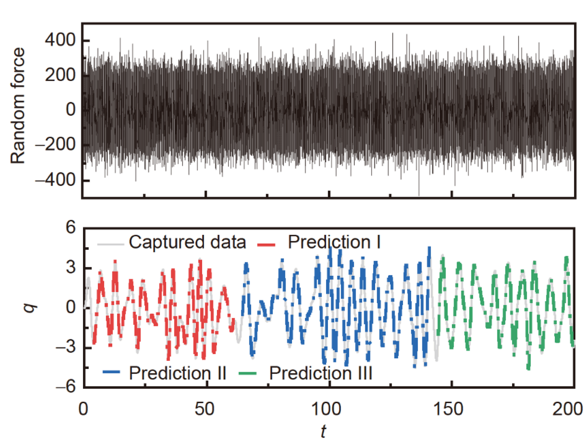


**Figure 3** (Color online) Demonstration of the switched Duffing oscillator with random excitation. Top row: captured discrete data of random excitation; bottom row: comparison of predicted trajectory and captured data.

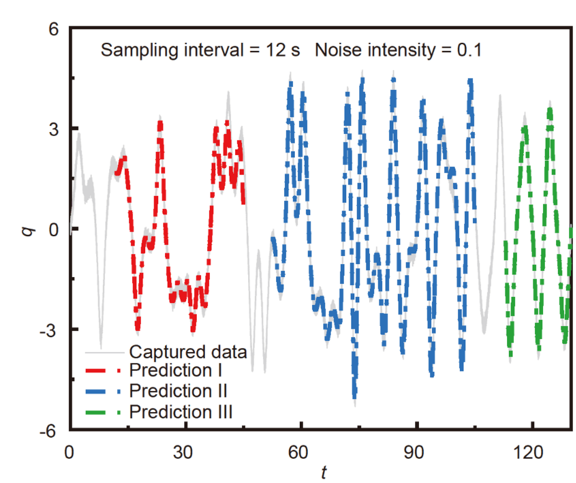


**Figure 5** (Color online) Comparison of predicted trajectory to captured data for switched Duffing oscillator with sinusoidal excitation (noise intensity 0.1, and integral interval 12 s).

## 4 Numerical example 2: Switched van der Pol system

Here, the van der Pol system with structure switching is adopted as the second example, with the objective to demonstrate the ability of the data-driven method to rapidly judge structure switches and reidentify the variational equation of motion. The classical van der Pol system stemming from the electric field is generally known for the existence of limit circle [30]. Here, we set the switched van der Pol system changing its structural form abruptly, that is, switching from limit circle to focal point, vice versa.

The differential equation of motion of van der Pol system is

$$L\ddot{q}+\alpha(q^2-a)\dot{q}+\frac{q}{C}=V, \tag{9}$$

and the associated integral-variational equation is

$$\int_{T_1}^{T_2} L\dot{q}\delta\dot{q}+\left(-V+\frac{1}{C}q+\alpha q^2\dot{q}-\alpha a\dot{q}\right)\delta q\,\mathrm{d}t=0, \tag{10}$$

where $L$ is the inductor, $C$ is the capacitor, $a$ is a threshold, $\alpha$ is the strength for energy injection/dissipation, and $V$ is the voltage. System parameters change abruptly and randomly which induce the changing of response characteristics, as shown in the first and second rows in Figure 6. According to the standard procedure mentioned above, the base functions of integrands are set as polynomials of state variables and external excitation. The integral-variational equation with undetermined coefficients is expressed as

$$\int_{T_1}^{T_2}(A_1+A_2\dot{q})\delta\dot{q}+(B_1+B_2F+B_3q$$
$$+B_4\dot{q}+B_5\dot{q}^2+B_6q^2+B_7q\dot{q}+B_8\dot{q}^3+B_9q^3$$
$$+B_{10}q^2\dot{q}+B_{11}q\dot{q}^2\Big)\delta q\,\mathrm{d}t=0. \tag{11}$$

We do not include the prior knowledge (e.g., dimensions of system parameters) in the selection of base functions, thus the integral-variational eq. (11) for van der Pol system includes more base functions and more undetermined coefficients compared to the integral-variational eq. (6) for Duffing oscillator.

For the case with sinusoidal excitation, the captured discrete data, the identified integral-variational equation for each stage, and the comparison of the predicted phase diagram and the observed phase diagram are shown in the second row, third row and fourth row in Figure 6, respectively. The tracking of switched van der Pol system with random excitation is shown in Figure 7. The data-driven method identifies the integral-variational equation rapidly and successfully tracks the van der Pol system with structure form switching. The relation of the relative error to the noise intensity is depicted in Figure 8, for several different integral intervals, 16, 20, 24, 30 and 40 s. The comparison of the captured data with Gaussian white noise with noise intensity 0.05 to the predicted trajectory of the data-driven method with integral interval 24 s is shown in Figure 9, which corresponds to the asterisked point in Figure 8. Similar con-

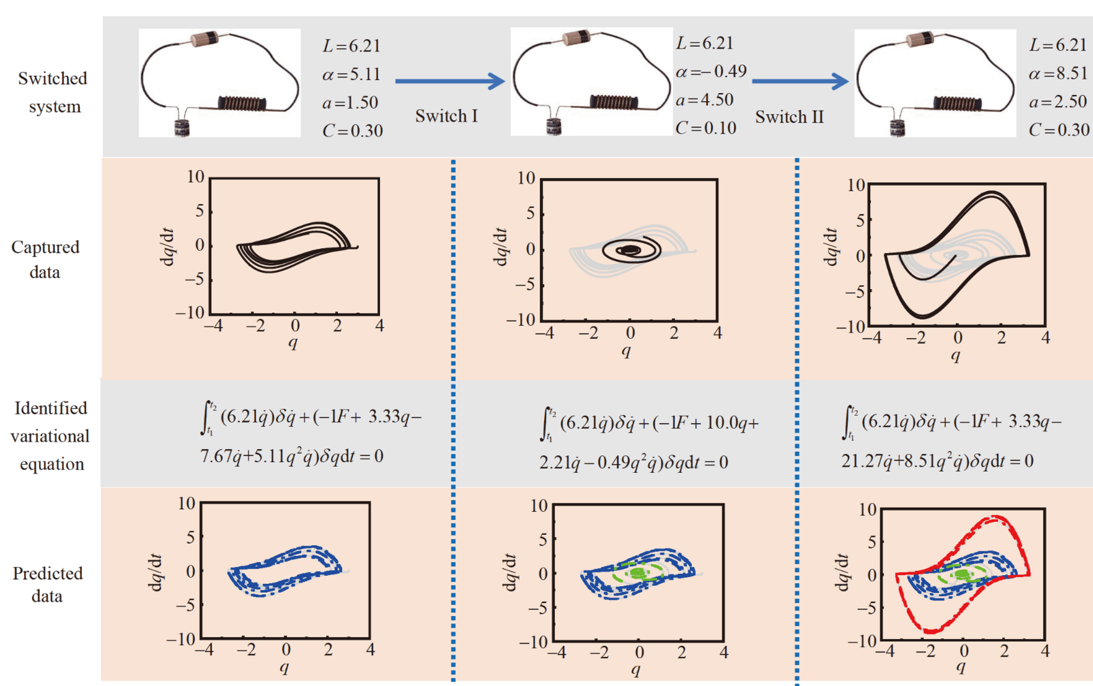


**Figure 6** (Color online) Demonstration of the switched van der Pol system with sinusoidal excitation. First row: physical model of van der Pol system with preset values of system parameters and moments of switches; second row: captured phase diagram with sampling frequency 0.001 s; third row: identified variational equation for each stage; fourth row: comparison of predicted phase diagram and captured phase diagram.

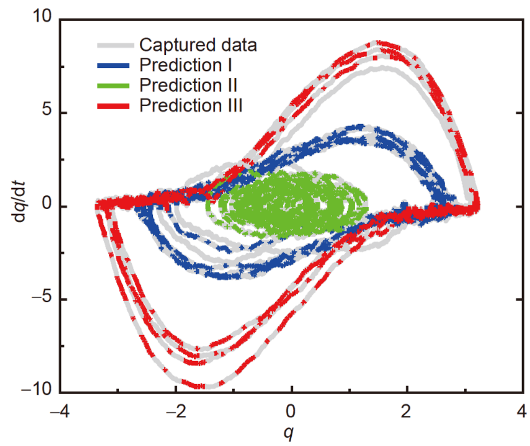


**Figure 7** (Color online) Comparison of predicted phase diagram with captured phase diagram for switched van der Pol system with random excitation.

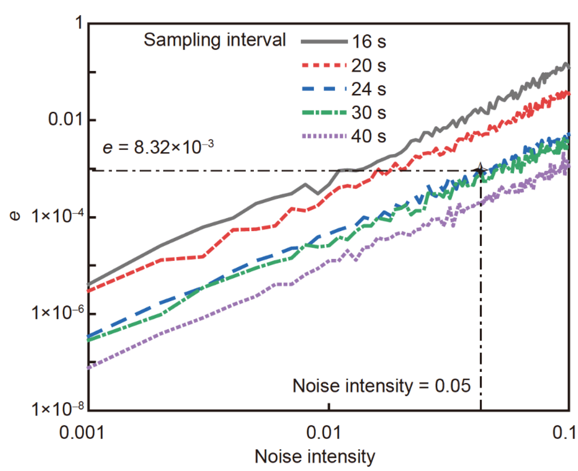


**Figure 8** The relation of the relative error of van der Pol system with sinusoidal excitation to noise intensity for five different integral intervals.

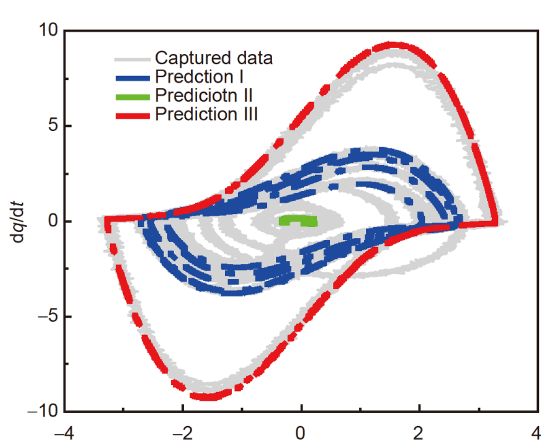


**Figure 9** (Color online) Comparison of predicted phase diagram to captured phase diagram for switched van der Pol system with sinusoidal excitation (noise intensity 0.05, and integral interval 24 s).

clusions are drawn as those for switched Duffing oscillator.

## 5 Conclusions

A data-driven method in variational framework was proposed for rapidly identifying switched systems from the discrete data captured from numerical simulations or experimental tests. It consists of several successively steps, i.e., capturing and recording the discrete data by various sensors, identifying the integral-variational equation of the initial stage, detecting deviation, and reidentifying the integral-variational equations of switched stages. The identifying procedure only needs a small amount of data due to the compact form of variational description, and is robustness to data noise due to the holistic viewpoint of variational description. Two representative examples were adopted to demonstrate its application, efficacy and robustness to data noise.

This data-driven method successfully tracks the abrupt and random switches of system parameters and structure forms. Thus, by combining with real-time control strategies, we can establish real-time control strategies of nonlinear systems by identifying the integral-variational equation of motion and calculating the feedback control through existing strategies (e.g., model predictive control [33,34] and direct control [35]). Some limitations of this data-driven method exist which leave room for future improvements. The numbers of the base functions of integrands increase dramatically with system dimension, which induces large computational cost for calculating undetermined coefficients. In addition, although this data-driven method tracks system switching with small time delay, it will fail if the system switches too rapidly, e.g., parameters or structural forms switches within half natural period of the associated linear system. For the switched systems with extremely high switching speed (with the totally random systems as limitation), there is not enough time remaining to captured discrete data, detecting deviation and reidentifying integral-variational equation timely.

*WANG Yong was supported by the National Natural Science Foundation of China (Grant Nos. 11872328 and 11472240). HUANG ZhiLong was supported by the National Natural Science Foundation of China (Grant Nos. 11532011 and 11621062). JIANG HanQing acknowledges Professor GUANG Biao of Zhejiang University.*

1 An J, Yan M, Zhou W, et al. Aircraft dynamic response to variable wing sweep geometry. J Aircraft, 1988, 25: 216–221
2 Yan B, Li Y, Dai P, et al. Aerodynamic analysis, dynamic modeling, and control of a morphing aircraft. J Aerosp Eng, 2019, 32: 04019058
3 Wang Y, Jin X, Huang Z. Stochastic averaging for quasi-integrable Hamiltonian systems with variable mass. J Appl Mech, 2014, 81: 051003
4 Cherniakov M. An Introduction to Parametric Digital Filters and Oscillators. Chichester: Wiley, 2003

5 Brunton S L, Kutz J N. Data-Driven Science and Engineering: Machine Learning, Dynamical Systems, and Control. Cambridge: Cambridge University Press, 2019
6 Goodfellow I, Bengio Y, Courville A. Deep Learning. Cambridge: The MIT Press, 2016
7 Bishop CM. Pattern Recognition and Machine Learning. New York: Springer-Verlag, 2006
8 Chu S R, Shoureshi R, Tenorio M. Neural networks for system identification. IEEE Control Syst Mag, 1990, 10: 31–35
9 Nelles O. Nonlinear System Identification: From Classical Approaches to Neural Networks and Fuzzy Models. New York: Springer-Verlag, 2001
10 Chen S, Billings S A, Grant P M. Non-linear system identification using neural networks. Int J Control, 1990, 51: 1191–1214
11 Kalman R E. A new approach to linear filtering and prediction problems. J Basic Eng, 1960, 82: 35–45
12 Juang J N, Pappa R S. An eigensystem realization algorithm for modal parameter identification and model reduction. J Guidance Control Dyn, 1985, 8: 620–627
13 Akaike H. Fitting autoregressive models for prediction. Ann Inst Stat Math, 1969, 21: 243–247
14 Bongard J, Lipson H. Automated reverse engineering of nonlinear dynamical systems. Proc Natl Acad Sci USA, 2007, 104: 9943–9948
15 Schmidt M, Lipson H. Distilling free-form natural laws from experimental data. Science, 2009, 324: 81–85
16 Quade M, Abel M, Shafi K, et al. Prediction of dynamical systems by symbolic regression. Phys Rev E, 2016, 94: 012214
17 Brunton S L, Proctor J L, Kutz J N. Discovering governing equations from data by sparse identification of nonlinear dynamical systems. Proc Natl Acad Sci USA, 2016, 113: 3932–3937
18 Wang W X, Yang R, Lai Y C, et al. Predicting catastrophes in nonlinear dynamical systems by compressive sensing. Phys Rev Lett, 2011, 106: 154101
19 Schaeffer H, Tran G, Ward R. Learning dynamical systems and bifurcation via group sparsity..
20 Mangan N M, Kutz J N, Brunton S L, et al. Model selection for dynamical systems via sparse regression and information criteria. Proc R Soc A, 2017, 473: 20170009
21 Quade M, Abel M, Kutz J N, et al. Sparse identification of nonlinear dynamics for rapid model recovery. Chaos, 2018, 28: 063116
22 Kaiser E, Kutz J N, Brunton S L. Sparse identification of nonlinear dynamics for model predictive control in the low-data limit. Proc R Soc A, 2018, 474: 20180335
23 Huang Z, Tian Y, Li C, et al. Data-driven automated discovery of variational laws hidden in physical systems. J Mech Phys Solids, 2020, 137: 103871
24 Mukhopadhyay S C, Jayasundera K P, Postolache O A. Modern Sensing Technologies. Cham: Springer, 2019
25 Whittaker E T. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies with an Introduction to the Problem of Three Bodies. London: Cambridge University Press, 1917
26 Lurie A I. Analytical Mechanics. Berlin: Springer-Verlag, 2002
27 Ben-Israel A, Greville T N E. Generalized Inverses: Theory and Applications. New York: Springer, 2003
28 Ott E. Chaos in Dynamical Systems. New York: Cambridge University Press, 2002
29 Schöll E, Schuster H G. Handbook of Chaos Control. Weinheim: Wiley-VCH, 2007
30 Nayfeh A H, Mook D T. Nonlinear Oscillations. New York: John Wiley, 1979
31 Bhaskar R, Nigam A. Qualitative physics using dimensional analysis. Artificial Intell, 1990, 45: 73–111
32 Lin Y K, Cai G Q. Probabilistic Structural Dynamics: Advanced Theory and Application. New York: McGraw-Hill. 1995
33 Camacho E F, Bordons C. Model Predictive Control in the Process Industry. London: Springer, 1995
34 Grüne L, Pannek J. Nonlinear Model Predictive Control: Theory and Algorithms. London: Springer, 2016
35 Yang Y, Wang Y, Jin X, et al. Stochastic averaging-based direct method for response control of nonlinear vibrating system. J Eng Mech, 2019, 145: 04019015